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FIXED POINTS ON TRANSVERSAL PROBABILISTIC SPACES

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Abstract. In this paper we introduce a notion of the probabilistic contraction on a lower transversal probabilistic space and prove two fixed point statements. The lower transversal probabilistic spaces are a natural extension of Menger's and probabilistic spaces.

1. Introduction and definitions

Transversal probabilistic spaces were introduced in 1998 by Tasković [6] in the following sense.

Let X be a nonempty set. The function $\rho : X \times X \to [0,1]$ is called an **upper probabilistic transverse** on X (or upper probabilistic transversal) if: $\rho[x,y] = \rho[y,x]$, and if there is a function $g : [0,1] \times [0,1] \to [0,1]$ such that

$$\text{(A)} \qquad \rho[x,y] \leq \max\Big\{\rho[x,z], \rho[z,y], g\left(\rho[x,z], \rho[z,y]\right)\Big\}$$

for all $x, y, z \in X$. An **upper transversal probabilistic space** is a set X together with a given upper probabilistic transverse on X. The function $g : [0,1] \times [0,1] \to [0,1]$ in (A) is called **upper (probabilistic) bisection function**.

In connection with this, the function $\rho : X \times X \to [0,1]$ is called a **lower probabilistic transverse** on X (or lower probabilistic transversal) if: $\rho[x,y] = \rho[y,x]$ and if there is a **lower (probabilistic) bisection function** $d : [0,1] \times [0,1] \to [0,1]$ such that

$$\text{(Am)} \qquad \min\Big\{\rho[x,z], \rho[z,y], d\left(\rho[x,z], \rho[z,y]\right)\Big\} \leq \rho[x,y]$$

for all $x, y, z \in X$. A **lower transversal probabilistic space** is a set X together with a given lower probabilistic transverse on X.

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Otherwise, a **transversal probabilistic space** is an upper and a lower transversal probabilistic space simultaneous.

The preceding definitions suggest that all probabilistic transverses $\rho : X \times X \to [0,1]$ may be interpreted as probability, that is to say, given any two points x and y of a transversal (upper or lower) probabilistic space X, rather than consider a single non-negative real number $\rho[x,y] \in [0,1]$ as a measure of the probabilistic transverse between x and y in X.

As an important example of lower transversal probabilistic spaces we have a Menger's (probabilistic) space.

K. Menger introduced in 1928 and 1942 the notion of probabilistic metric space. O. Kaleva and S. Seikkala proved in 1984 that each Menger's space, which is a special probabilistic metric space, can be considered as a fuzzy metric space.

In further, we shall denote the distribution function $F(p,q)$ by $F_{p,q}$ and $F_{p,q}(x)$ will represent the value of $F_{p,q}$ at $x \in \mathbb{R}$.

The function $F_{p,q}(p,q \in X)$ are assumed to satisfy the following conditions: $F_{p,q}(x) = 1$ for $x > 0$ iff $p = q$, $F_{p,q}(0) = 0$ and $F_{p,q} = F_{q,p}$ for all $p, q \in X$. In further let $\rho[u,v] = F_{u,v}(x)$ with the preceding conditions.

In the theory of metric spaces, as and in the lower transversal probabilistic spaces, it is extremely convenient to use a geometrical language inspired by classical geometry.

Thus elements of a lower transversal probabilistic space will usually be called **points**. Given a lower transversal probabilistic space (X,ρ), with the lower bisection function $d : [0,1] \times [0,1] \to [0,1]$ and a point $a \in X$, the **open ball**, in notation $d(B(a,r))$, of center a and radius $r > 0$ is the set

$$d\big(B(a,r)\big) = \Big\{x \in X : \rho[a,x] > r\Big\}.$$

The concept of a neighborhood in a lower transversal probabilistic space X for the lower probabilistic transverse $\rho[p,q] := F_{p,q}(x)$ is the following. If $p \in X$, and μ, σ are positive reals, then an (μ,σ) - **neighborhood** of p, denoted by $U_p(\mu,\sigma)$, is defined by

$$U_p(\mu,\sigma) = \{q \in X : \rho[p,q] = F_{p,q}(\mu) > 1 - \sigma\}.$$

The above topology satisfies the first axiom of countability. In this topology a sequence $\{p_n\}_{n\in\mathbb{N}}$ in X converges to a point $p \in X$ (in notation $p_n \to p$) if and only if for every $\mu > 0$ and $\sigma > 0$, there exists an integer $M(\mu,\sigma)$ such that $p_n \in U_p(\mu,\sigma)$, i. e., $\rho[p,p_n] > 1 - \sigma$ whenever $n \geq M(\mu,\sigma)$. The sequence $\{p_n\}_{n\in\mathbb{N}}$ will be called fundamental in X if for each $\mu > 0$, $\sigma > 0$ there is an integer $M(\mu,\sigma)$ such that $\rho[p_n,p_m] = F_{p_n,p_m}(\mu) > 1 - \sigma$ whenever $n, m \geq M(\mu,\sigma)$. In analogy with the completion concept of metric space, a lower transversal probabilistic space X will be called **complete** if each fundamental sequence in X converges to an element in X.

2. Main results

In further we introduce a notion of a probabilistic contraction on a lower transversal probabilistic space and prove a fixed point theorem.

A mapping T of a lower transversal probabilistic space (X, ρ) into itself for $\rho[u,v] = F_{u,v}(x)$ will be called a **probabilistic contraction** if there exists a nondecreasing function $\varphi : \mathbb{R}^{\circ}_{+} \to \mathbb{R}^{\circ}_{+} := [0, +\infty)$ such that

$$\text{(As)} \qquad \lim_{n\to\infty} \varphi^n(t) = +\infty \quad \text{for every } t > 0$$

and such that

$$\text{(Pc)} \qquad F_{Tu,Tv}(x) \geq \min\Big\{F_{u,v}(\varphi(x)), F_{u,Tu}(\varphi(x)), F_{v,Tv}(\varphi(x)), F_{u,Tv}(\varphi(x)), F_{v,Tu}(\varphi(x))\Big\}$$

for all $u, v \in X$ and for every $x \in \mathbb{R}_+ := (0, +\infty)$.

The function $d : [0,1]\times[0,1] \to [0,1]$ is **nondecreasing** if $a_i, b_i \in \in [0,1]$ and $a_i \leq b_i$ $(i = 1,2)$ implies $g(a_1, a_2) \leq g(b_1, b_2)$. Now we shall prove the following result for the preceding class of functions (Pc) on a special lower transversal probabilistic space.

Theorem 1. *Let (X, ρ) be a complete lower transversal probabilistic space, where the lower transverse $\rho[u,v] = F_{u,v}$ and the lower bisection function $d : [0,1] \times [0,1] \to [0,1]$ is nondecreasing such that $d(t,t) \geq t$ for every $t \in \mathbb{R}_+$. If T is any probabilistic contraction mapping of X into itself, then there is a unique point $p \in X$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in X$.*

Proof. For this proof the following inequalities are essential. Namely, from the conditions for the function $d : [0,1]^2 \to [0,1]$ we obtain the following inequalities

$$\text{(1)} \qquad d(a,b) \geq d(\min\{a,b\}, \min\{a,b\}) \geq \min\{a,b\}$$

for all $a, b \in [0,1]$. On the other hand, since X is a lower transversal probabilistic space, for every $x \geq 0$ we have the following inequalities

$$\text{(2)} \qquad \begin{aligned} F_{a,b}(x) &\geq \min\Big\{F_{a,c}(x), F_{c,b}(x), d\left(F_{a,c}(x), F_{c,b}(x)\right)\Big\} \\ &\geq \min\Big\{F_{a,c}(x), F_{c,b}(x)\Big\} \end{aligned}$$

To prove the existence of the fixed point, consider an arbitrary $u \in \in X$, and define $u_n = T^n(u)$, for $n \in \mathbb{N} \cup \{0\}$. We show that the sequence $\{u_n\}_{n\in\mathbb{N}\cup\{0\}}$ is fundamental in X. Then for $a \in \mathbb{R}_+$ and $m > n$ $(m, n \in \mathbb{N})$ from (2) is

$$\text{(3)} \qquad F_{u_n,u_m}(a) \geq \min\Big\{F_{u_n,u_{n+1}}(a), \dots, F_{u_{m-1},u_m}(a)\Big\}$$

On the other hand, since T is a probabilistic contraction mapping, from (2) we obtain the folowing inequalities

$$\begin{aligned} F_{u_n,u_{n+1}}(a) &= F_{Tu_{n-1},Tu_n}(a) \geq \\ &\geq \min\left\{F_{u_{n-1},u_n}(\varphi(a)), F_{u_n,u_{n+1}}(\varphi(a)), F_{u_{n-1},u_{n+1}}(\varphi(a))\right\} \geq \\ &\geq \min\left\{F_{u_{n-1},u_n}(\varphi(a)), F_{u_n,u_{n+1}}(\varphi(a))\right\} \end{aligned} \tag{4}$$

and thus

$$F_{u_n,u_{n+1}}(\varphi(a)) \geq \min\left\{F_{u_{n-1},u_n}(\varphi^2(a)), F_{u_n,u_{n+1}}(\varphi^2(a))\right\} \tag{5}$$

From (4) and (5) it folows by induction that for every integer $k \in \mathbb{N}$ the following inequality holds

$$F_{u_n,u_{n+1}}(a) \geq \min\left\{F_{u_{n-1},u_n}(\varphi(a)), F_{u_n,u_{n+1}}\left(\varphi^k(a)\right)\right\},$$

that is, when $k \to +\infty$, we obtain $F_{u_n,u_{n+1}}(a) \geq F_{u_{n-1},u_n}(\varphi(a))$ for every $n \in \mathbb{N}$, i. e.,

$$F_{u_n,u_{n+1}}(a) \geq F_{u_0,u_1}(\varphi^n(a))$$

for every $n \in \mathbb{N}$. Hence, from the former inequality (3), we obtain

$$F_{u_n,u_m}(a) \geq \min\left\{F_{u_0,u_1}(\varphi^n(a)), F_{u_0,u_1}(\varphi^{n+1}(a)), \ldots, F_{u_0,u_1}(\varphi^{m-1}(a))\right\}$$

that is, $F_{u_n,u_m}(a) \geq F_{u_0,u_1}(\varphi^n(a))$. Hence, $\{u_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a fundamental sequence in X. Since X is a complete space, there is an $p \in X$ such that $u_n \to p$, that is $T^n(u) \to p$. Then, from the former facts, we have

$$F_{Tp,p}(a) \geq F_{Tp,p}(\varphi(a))$$

for every $a \in \mathbb{R}_+$, that is $Tp = p$. We further prove the uniqueness. Suppose $p \neq q$ and $Tp = p$, $Tq = q$. Then, there exists an $x > 0$ and an $0 \leq a < 1$, such that $F_{p,q}(x) = a$. However, since T is a probabilistic contraction mapping, for each $n \in \mathbb{N}$ we have

$$a = F_{p,q}(x) = F_{Tp,Tq}(x) \geq \cdots \geq F_{p,q}(\varphi^n(x)),$$

and hence, since $F_{p,q}(\varphi^n(x)) \to 1$ as $n \to \infty$, it follows that $a = 1$. This contradicts the choice of $0 \leq a < 1$, and therefore, the fixed point is unique. The proof is complete.

In connection with the preceding statement, from our the Principle of Symmetry (see: Tasković, *Math. Japonica*, **35** (1990), p. 661), we obtain as an immediate consequence of Theorem 1 the following result.

Theorem 2. *Let (X,ρ) be a complete lower transversal probabilistic space, where the lower transverse $\rho[u,v]=F_{u,v}$ and the lower bisection function $d:[0,1]\times[0,1]\to[0,1]$ is nondecreasing such that $d(t,t)\geq t$ for every $t\in\mathbb{R}_+$. If there exists a nondecreasing function $\varphi:\mathbb{R}_+^\circ\to\mathbb{R}_+^\circ$ such that (As) and if there is a function $n:X\to\mathbb{N}$ such that*

$$F_{T^{n(u)}(u),T^{n(v)}(v)}(x)\geq\min\Big\{F_{u,v}\left(\varphi(x)\right),F_{u,T^{n(u)}u}\left(\varphi(x)\right),$$
$$F_{v,T^{n(v)}v}\left(\varphi(x)\right),F_{u,T^{n(v)}v}\left(\varphi(x)\right),F_{v,T^{n(u)}u}\left(\varphi(x)\right)\Big\}$$

for all $u,v\in X$ and for every $x\in\mathbb{R}_+$, then T has exactly one fixed point $p\in X$ and $T^nq\to p$ for every $q\in X$.

3. Some consequences

Let B denote the set of all Δ-norms. A **Menger space** is a triplet (E,F,Δ), where (E,F) is a probabilistic metric space and $\Delta\in B$ satisfies the following inequality:

$$F_{p,r}(x+y)\geq\Delta\left(F_{p,q}(x),F_{q,r}(y)\right)$$

for all p, q, $r\in E$ and for all $x,y\geq 0$. Metric spaces are special cases of Menger spaces with $\Delta(x,x)\geq x$ for every $x\in[0,1]$.

If we chosen a lower bisection function $d:[0,1]^2\to[0,1]$ such that $d=\Delta$ (for $\Delta\in B$), then we obtain immediate that every Menger's space, for $\rho[x,y]=F_{x,y}$, is a lower transversal probabilistic space.

Hence, Theorems 1 and 2 immediate hold and for Menger's spaces. Also, since every probabilistic (metric) space is a lower transversal probabilistic space, hence Theorems 1 and 2 hold and for probabilistic spaces; similar, and for fuzzy metric spaces.

On the other hand, as an immediate consequence of the preceding Theorem 1 we obtain directly the following result on Menger's spaces.

Corollary 1. (Bylka [1]). *Let (E,F,Δ) be a complete probabilistic Menger space, where Δ is a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$, and T a mapping of E into itself. If $\varphi:\mathbb{R}_+^\circ\to\mathbb{R}_+^\circ$ is a nondecreasing function such that (As) and*

$$F_{Tu,Tv}(x)\geq F_{u,v}\left(\varphi(x)\right)$$

for $x\in\mathbb{R}_+$ and for all $u,v\in E$, then T has a unique fixed point $p\in E$ and $T_{T^nq,p}(x)\to 1$ for every $q\in E$ and $x\in\mathbb{R}_+$.

4. Open problems

We notice that are preceding results of this note (Theorems 1 and 2) given for a special probabilistic transverse in case $\rho[u, v] = F_{u,v}(x)$.

In connection with this, formulate some new statements of the preceding type for arbitrary upper or lower probabilistic transverses!

Formulate and a correspond statement (analogous to Theorem 1) for upper transversal probabilistic spaces!

5. References

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